§ 3.5 (cont'd)

Rule 2 Method of Undetermined Coefficients:

if $f(x)$ is in form $\mathbb{P}_m e^{rx} \cos kx$ or $\sin kx$

use trial y_p:

$$x^s\left[(A_0+A_1x+A_2x^2+\dots+A_mx^m)e^{rx}\cos kx+(B_0+B_1x+B_2x^2+\dots+B_mx^m)e^{rx}\sin kx\right]$$

where s is smallest integer s.t. no term in y_p duplicates a term in y_c

§3.8 Endpoint Problems and Eigenvalues

recall: $y''+p(x)y'+q(x)y=0$ w $y(a)=0$ and $y'(a)=0$

only sol'n is trivial: $y(x)\equiv 0$

now explore $y''+p(x)y'+q(x)y=0$ w $y(a)=0$ and $y(b)=0$

where $a<b$

find sol'n on open $I=(a,b)$

called boundary value problems

ex. consider $y''+3y=0$ with $y(0)=0$ and $y(\pi)=0$

char eqn: $r^2+3=0$

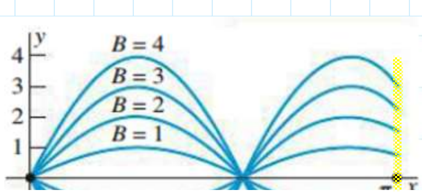
$r^2=-3$

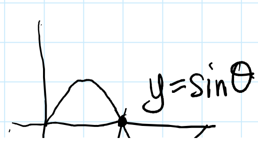
$r=\pm\sqrt{3}\,i$

gen soln: $y(x)=e^{0x}(A\cos\sqrt{3}x+B\sin\sqrt{3}x)$

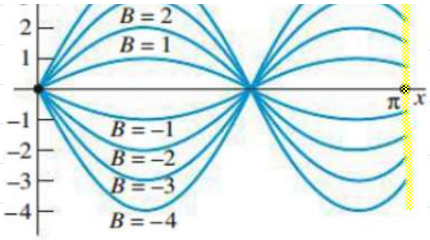
$y(0)=A\,(1)+0=0\Rightarrow A=0$

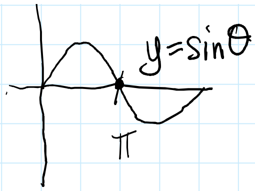
$y(\pi)=B\underbrace{\sin(\sqrt{3}\pi)}_{(-0.7458)}=0$





no value of B will make this zero





no value of B will make this zero at $x = \pi$

conclude: only sol'n is $y \equiv 0$ (trivial)

ex. consider $y'' + 4y = 0$ w $y(0) = 0$ and $y(\pi) = 0$

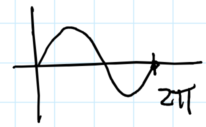
$r = \pm 2i$

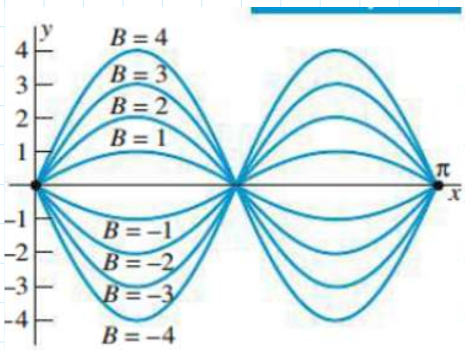
gen soln: $y(x) = A\cos 2x + B \sin 2x$

again $y(0) = A = 0$

now $y(\pi) = 0 = B \sin 2\pi$

$0 = B\ (0)$





zero regardless of value of B

conclude: infinitely many non-trivial sol'ns

Eigenvalue Problems

expand format of BVP to: $y'' + p(x)y' + \lambda q(x) y = 0$ $\quad y(a) = 0$, $y(b) = 0$

eigenvalue

in prev. examples: $y'' + 3y = 0$ $\quad y(0) = 0,\ y(\pi) = 0$

consider $q(x) = 1$
$\lambda = 3$

$y'' + 4y = 0$ $\quad y(0) = 0,\ y(\pi) = 0$

$q(x) = 1$
$\lambda = 4$

in an eigenvalue problem, question is posed;
for what values of λ does there exist a non-trivial sol'n of BVP?

generalize: $y'' + \lambda y = 0$ where $y(0)=0$, $y(\pi)=0$

saw that $\lambda = 3$ ← not an eigenvalue

generalize $y'' + p(x)y' + \lambda q(x) y = 0$ where $y(a)=0$, $y(b)=0$:

suppose λ_* is an eigenvalue and $y_*(x)$ is non-trivial sol'n

becomes $y_*'' + p(x)y_*' + \lambda q(x) y_* = 0$ where $y_*(a)=0$, $y_*(b)=0$

call y_* an eigenfunction associated with λ_*

ex. recall $y'' + 4y = 0$ $\lambda_* = 4$

y_* is any constant multiple of $\sin 2x$

can call HG since constant multiple of eigenfunction is also an eigenfunction

i.e., if $y = y_*(x)$ satisfies $y'' + 4y = 0$ w $\lambda = \lambda_*$
then so does any $\underset{\text{constant multiple}}{c}\, y_*(x)$

ex. determine eigenvalue(s) and associated eigenfunction(s) for BVP
$y'' + \lambda y = 0$ where $y(0)=0$ and $y(L)=0$ $L>0$

case 1: if $\lambda = 0$: $y'' = 0 \Rightarrow$ gen soln: $y(x) = Ax + B$ ← since y''

$y(0) = B = 0$

if $y(0)=y(L)=0$ then $A=B=0$

$y(L) = A(L) = 0 \Rightarrow A = 0$

L>0 by def'n   conclude

$\therefore \lambda = 0$ is not an eigenvalue

case 2: if $\lambda < 0$ write λ as $-\alpha^2$ where $\alpha > 0$

rewrite: $y'' - \alpha^2 y = 0$

char. eqn: $r^2 - \alpha^2 = 0 \Rightarrow r = \pm\alpha$

gen soln: $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

gen soln: $y(x) = c_1 e^{\cdots} + c_2 e^{\cdots}$

rewrite ITO $\cosh x$, $\sinh x$ $= A\cosh(\alpha x) + B\sinh(\alpha x)$

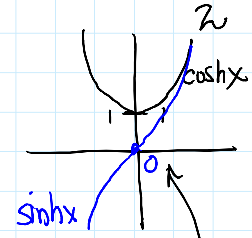
$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$A = c_1 + c_2$ $\qquad B = c_1 - c_2$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

when $y(0) = 0 = A\cosh(0) + B\sinh(0)$ $\qquad \cosh(0) \to 1$, $\sinh(0) \to 0$

$0 = A$



when $y(L) = 0 = B\sinh(\alpha L)$

by def'n $\alpha \neq 0$

also sinhx only zero once

B must be zero

conclude: NO negative eigenvalues

case 3: if $\lambda > 0$ write as $\lambda = \alpha^2$ where $\alpha > 0$

rewrite: $y'' + \alpha^2 y = 0$

char eqn. $r^2 + \alpha^2 = 0$

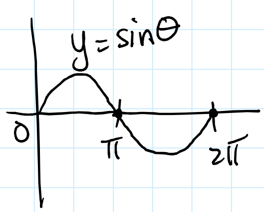
$r^2 = -\alpha^2$

$r = \pm \alpha i$

gen soln: $y(x) = e^{0x}(A\cos\alpha x + B\sin\alpha x)$

$y(0) = 0 = A$

$y(L) = 0 = B\sin(\alpha L)$ $\qquad \alpha > 0$, $L > 0$



in this scenario, it's possible for $B \neq 0$

because $\sin(\alpha L) = 0$ when $\alpha L = \pi, 2\pi, \dots, n\pi, \dots$ $\qquad n = 1, 2, 3, \dots$

since $\lambda = \alpha^2$

square both sides

$\alpha^2 L^2 = \pi^2, (2\pi)^2, \dots, (n\pi)^2, \dots$

$\lambda = \dfrac{\pi}{L^2}, \dfrac{4\pi^2}{L^2}, \dots, \dfrac{n^2\pi^2}{L^2}, \dots$

conclude: the infinite set of eigenvalues (positive) can be written as $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ where $n = 1, 2, 3, \dots$

then $y_n(x) = B\sin(\alpha x)$ $\qquad$ since $\alpha^2 = \dots, \dfrac{n^2\pi^2}{L^2}, \dots$

then $y_n(x) = B\sin(\alpha x)$ since $\alpha^2 = \ldots, \frac{n^2\pi^2}{L^2}, \ldots$

$\alpha = \ldots, \frac{n\pi}{L}, \ldots$

$$y_n = B\sin\left(\frac{n\pi}{L}x\right) \text{ where } n = 1, 2, 3, \ldots$$

↑ set of eigenfunctions

since $\lambda > 0$ $\quad \lambda_1 < \lambda_2 < \lambda_3 < \ldots, < \lambda_n < \ldots \to +\infty$

↑ sequence of λ diverges

---

change init. cond: $\quad y'' + \lambda y = 0$ where $y(0) = 0$ $\quad y'(L) = 0$

Know from prev ex. that $\lambda > 0$ let $\lambda = \alpha^2$

then gen soln: $y(x) = A\cos\alpha x + B\sin\alpha x$

$y(0) = 0 = A \Rightarrow y(x) = B\sin\alpha x$

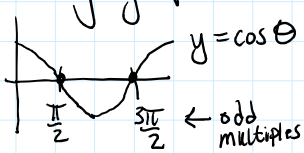
$y'(x) = B\alpha\cos\alpha x$

$y'(L) = 0 = B\alpha\cos(\alpha L)$

this time it's possible $B \neq 0$ when αL is ODD, positive multiple of $\frac{\pi}{2}$

this is where we stopped in lecture
I'll explain the rest here so you can digest it @ your leisure - it's very similar to example above

considering graph:



$\alpha L = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, \ldots$

$\alpha^2 L^2 = \frac{\pi^2}{4}, \frac{9\pi^2}{4}, \ldots, \frac{(2n-1)^2\pi^2}{4}, \ldots$

$\lambda = \alpha^2 = \ldots, \frac{(2n-1)^2\pi^2}{4L^2}, \ldots$

$\Downarrow$

$\alpha = \frac{(2n-1)\pi}{2L}$

nth eigenvalue $\quad$ nth associated eigenfunction

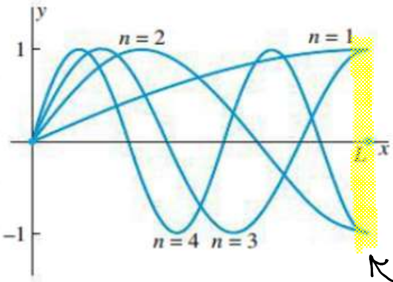
$$\therefore \lambda_n = \frac{(2n-1)^2\pi^2}{4L^2} \text{ and } y_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

again recall that coefficient of x

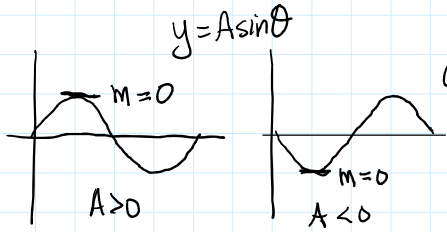
$4L^2$ $\qquad\qquad\qquad \left(\underbrace{\;2L\;} \right)$

again, recall that coefficient of x determines period

in particular, n impacts period (see figure to the left)



@ $x = L$ if given that $y'(L) = 0$ that means $m_{tan} = 0$ and it would follow that extrema of sine curve would occur @ L



again, referring to figure and thinking of a period of sine, we've shown that $x = L$ is location of middle of half-period